Property: If Q is orthogonal, then $(Qx)\cdot(Qy) = x\cdot y$ for all $x, y \in \mathbb{R}^n$

Proof: $(Qx)\cdot(Qy) = (Qx)^T(Qy) = x^T Q^T Q y = x^T I y = x^T y = x\cdot y$ ▨

Two consequences:

- $\|Qx\|^2 = (Qx)\cdot(Qx) = x\cdot x = \|x\|^2$. So $\|Qx\| = \|x\|$.
- Let θ be the angle between x and y

  $(Qx)\cdot(Qy) = x\cdot y = \|x\|\|y\|\cos\theta = \|Qx\|\|Qy\|\cos\theta$

  $\Rightarrow$ θ is also the angle between Qx and Qy

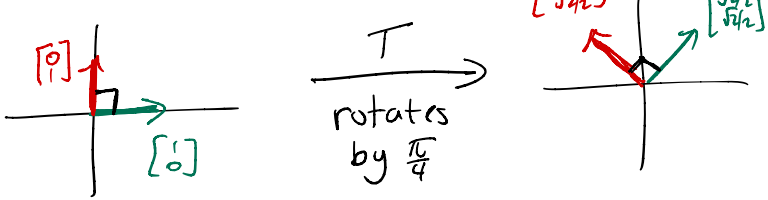
Connection to linear transformations

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation whose associated matrix Q is orthogonal.

Then by the above properties, T preserves length and angles:

i.e. $\|T(v)\| = \|v\|$ $\forall\, v \in \mathbb{R}^n$ and $\measuredangle(v,w) = \measuredangle(T(v), T(w))$

Ex: Let $Q = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$



Computing Upshot

Matrices with orthonormal columns are great for computing projections and least-squares solutions

Suppose a subspace W has orthonormal basis $\{q_1, \cdots, q_n\}$

Then $Q = [q_1 \cdots q_n]$ has orthonormal columns

The projection of b onto W is simply

$$\text{proj}_W b = Q(\underbrace{Q^TQ}_{=I})^{-1}Q^Tb = QQ^Tb. \quad \text{(No inverses required!)}$$

This can be expressed in terms of 1-dim'l projections:

$$QQ^Tb = [q_1 \cdots q_n]\begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} b$$

$$= [q_1 \cdots q_n]\begin{bmatrix} q_1 \cdot b \\ \vdots \\ q_n \cdot b \end{bmatrix}$$

$$= (q_1 \cdot b)q_1 + (q_2 \cdot b)q_2 + \cdots + (q_n \cdot b)q_n$$

$$= \text{proj}_{q_1} b + \text{proj}_{q_2} b + \cdots + \text{proj}_{q_n} b \quad \text{(since } q_i \cdot q_i = 1 \ \forall i\text{)}$$

So $\text{proj}_W b = \text{proj}_{q_1} b + \text{proj}_{q_2} b + \cdots + \text{proj}_{q_n} b$

where $\{q_1, \cdots, q_n\}$ is an orthonormal basis of W

Moreover: this formula also holds for any orthogonal basis of W

i.e. $\{c_1, \cdots, c_n\}$ orthogonal $\Rightarrow \text{proj}_W b = \text{proj}_{c_1} b + \cdots + \text{proj}_{c_n} b$ (HW

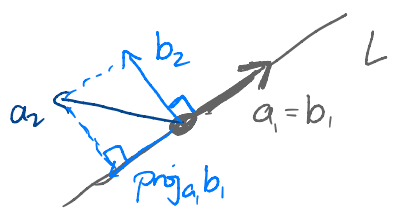
The Gram Schmidt Process

Having an orthormal basis is useful, but how do we find one? The Gram Schmidt process uses projections to create an orthonormal basis from a given basis.

Let W be a subspace of $\mathbb{R}^m$ with basis $\{a_1,\dots,a_n\}$. We will create an orthonormal basis $\{q_1,\dots,q_n\}$.

Step 1: Let $b_1 = a_1$

Step 2: Let $b_2 = a_2 - \text{proj}_{b_1} a_2$



Why? Since b_2 is simply the error vector, we know $b_2 \perp L$ and so $b_2 \cdot b_1 = 0$

Step i: Let $b_i = a_i - \text{proj}_{b_1} a_i - \text{proj}_{b_2} a_i - \cdots - \text{proj}_{b_{i-1}} a_i$

Why? Since $\{b_1,\dots,b_{i-1}\}$ is orthogonal,

$$\text{proj}_{b_1} a_i + \cdots + \text{proj}_{b_{i-1}} a_i = \text{proj}_{\text{span}\{b_1,\dots,b_{i-1}\}} a_i$$

So $b_i = a_i - \text{proj}_{\text{span}\{b_1,\dots,b_{i-1}\}} a_i$ is once again the error vector, which we know is perpendicular to $W = \text{Span}\{b_1,\dots,b_{i-1}\} \Rightarrow b_i \cdot b_j = 0 \quad \forall\, j < i$.

Upon termination, we have an orthogonal basis $\{b_1,\dots,b_n\}$. Now normalize (ie. $q_i = \frac{b_i}{\|b_i\|}\ \forall\, i$) to get an orthonormal basis $\{q_1,\dots,q_n\}$.

Ex: Consider the basis $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\2\\0\\1\end{bmatrix}\right\}$ (a_1, a_2, a_3) of the subspace W spanned by the basis vectors. Use Gram Schmidt to construct an orthonormal basis for W.

1. $b_1 = a_1$

2. $b_2 = a_2 - \text{proj}_{b_1} a_2 = a_2 - \frac{a_2 \cdot b_1}{b_1 \cdot b_1} b_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{2}{2}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix}$

3. $b_3 = a_3 - \text{proj}_{b_1} a_3 - \text{proj}_{b_2} a_3$

$$= a_3 - \frac{a_3 \cdot b_1}{b_1 \cdot b_1} b_1 - \frac{a_3 \cdot b_2}{b_2 \cdot b_2} b_2$$

$$= \begin{bmatrix}-1\\2\\0\\1\end{bmatrix} - \frac{-1}{2}\begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{3}{2}\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\\1/2\\-1/2\end{bmatrix}$$

Now $q_1 = \frac{b_1}{\|b_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\end{bmatrix}$

$q_2 = \frac{b_2}{\|b_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix}$

$q_3 = \frac{b_3}{\|b_3\|} = \begin{bmatrix}-1/2\\1/2\\1/2\\-1/2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}$

So $\left\{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}\right\}$ is an orthonormal basis